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A Note on Taylor Instability in Circular Couette Flow

The problem of Taylor instability is reexamined, and it is shown that the effect of geometry (radius ratio) can be very nearly suppressed by a proper choice of variables. The two cases of cylinders rotating in the same direction and in opposite directions are treated separately.

Introduction

SOME RECENT experiments at GALCIT on transition in circular Couette flow have been described in reference [1].¹ The small rotating-cylinder apparatus used in these experiments has now been handed over to a beginning laboratory course in fluid mechanics, and students are asked to find the position of the Taylor instability boundary for various combinations of cylinder rotation. In an effort to present the analytical background for this experiment with the greatest possible brevity and simplicity, the usual formulation of Taylor instability as an eigenvalue problem has been reexamined and reduced essentially to an

exercise in algebra. The simplified analysis is presented here because it leads to some new insights and because the method may not be limited to this one problem.

The subject of Taylor instability is commonly approached by way of an inviscid stability criterion due to Rayleigh [2]. He showed that in a steady, purely circulatory flow, with tangential velocity $\bar{v}(r)$, the radial equilibrium is unstable (i.e., a radial displacement tends to increase) if the square of the circulation decreases outward. Since $dr^2\bar{v}^2/dr = 2r^2\bar{\omega}\bar{\xi}$, where $\bar{\xi} = d\bar{v}/dr$ is vorticity and $\bar{\omega} = \bar{v}/r$ is angular velocity, this inviscid criterion for circulatory flow can also be expressed in the form

$$\bar{\xi}\bar{\omega} < 0 \quad (1)$$

In other words, the motion is unstable whenever the sense of the local rotation $\bar{\xi}$ is opposite to the sense of the general rotation $\bar{\omega}$.

The next step was taken by Taylor [3], who determined the effect of viscosity on the stability boundary both experimentally and theoretically. His experimental results for three combinations of cylinder radii are shown in Fig. 1. Taylor also found that (a) the secondary motion resulting from the instability is a set of axially symmetric cellular vortices; (b) the cells are nearly

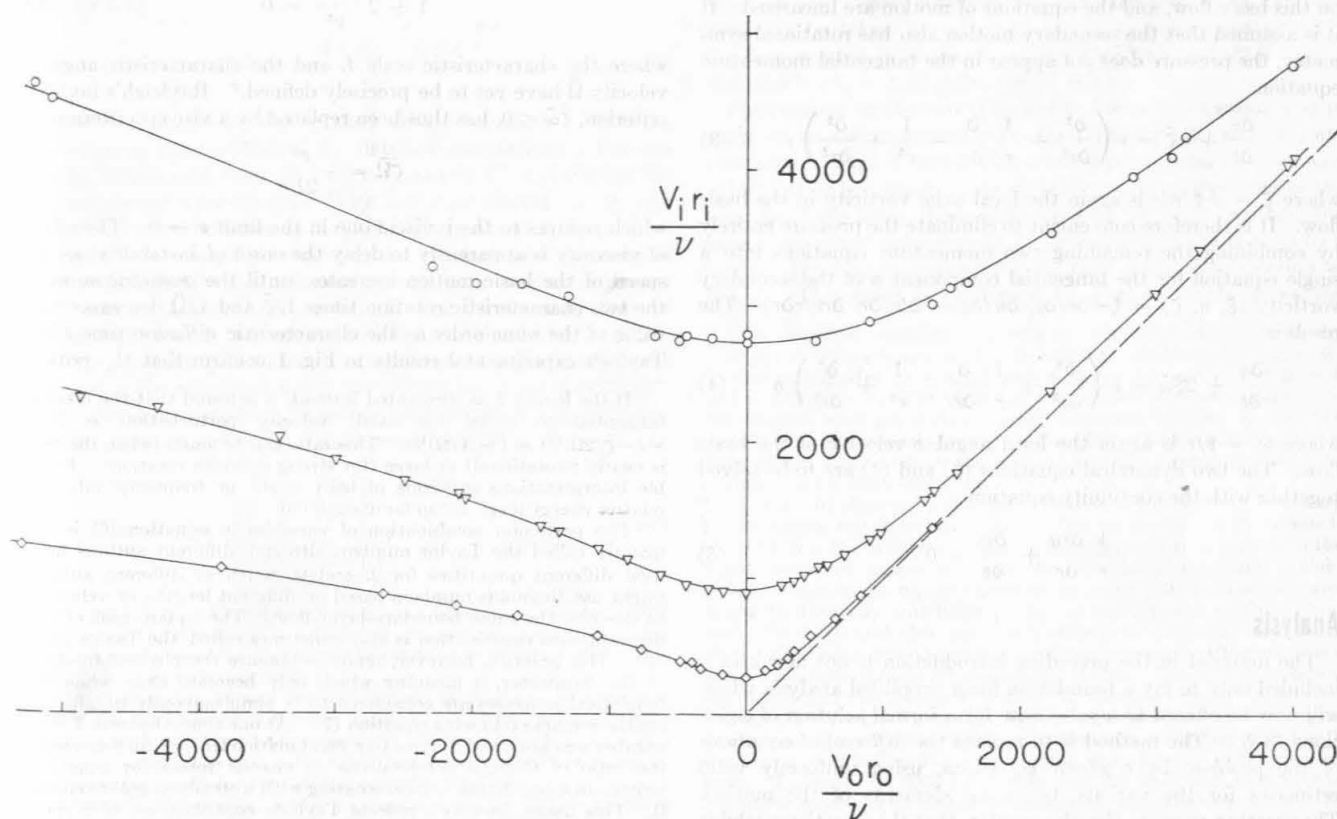


Fig. 1 Dashed line shows theoretical inviscid stability boundary according to Rayleigh [2]. Solid lines show experimental viscous stability boundary according to Taylor [3]. From top to bottom: Taylor's data are for $r_i/r_o = 0.942, 0.880, 0.743$.

square in cross section; and (c) the secondary motion is confined essentially to the part of the fluid which is unstable according to the inviscid criterion.

In any self-consistent discussion of stability for a viscous fluid, the basic flow $\bar{v}(r)$ should satisfy the viscous equations. This flow should therefore be taken, in a usual notation, as

$$\bar{v} = Ar + \frac{B}{r} \quad (2)$$

where the constants A and B are determined by the boundary condition of zero relative velocity at the two walls. The parameter A is a measure of vorticity and vanishes for potential flow; the parameter B is a measure of torque and vanishes for a solid-body rotation. There is no loss of generality in defining the angular velocity of the inner cylinder to be always positive, whereupon Rayleigh's criterion yields the inviscid stability boundary shown by the dashed lines in Fig. 1. The two segments of this boundary correspond to the vanishing of the two factors in equation (1). In the right quadrant, the cylinders rotate in the same direction, and the entire motion is either stable or unstable. The boundary is the line $\bar{\zeta} = 0$ (irrotational flow). In the left quadrant, the cylinders rotate in opposite directions, and there is always instability but only in the part of the fluid between the inner cylinder and the nodal radius $r = r_c = (-B/A)^{1/2}$ at which \bar{v} and $\bar{\omega}$ are zero. The inviscid boundary is formally the line $\omega_i = 0$ (inner cylinder at rest); however, in the sense that ω_i and $\bar{\omega}$ approach zero together, this is also the line $\bar{\omega} = 0$. Note that the parameter A in equation (2) (and hence also $\bar{\zeta} = 2A$) is negative, and B is positive, whenever Rayleigh's inviscid criterion indicates instability. The angular velocity $\bar{\omega}$ in the unstable part of the annulus is always positive.

For the purposes of this paper, a sufficient analytical formulation of Taylor's viscous stability problem is as follows. Suppose that the basic flow, denoted by $(\bar{u}, \bar{v}, \bar{w})$ in cylindrical polar coordinates (r, θ, z) , is steady and purely tangential. It follows that $\bar{u} = \bar{w} = 0$ and that $\bar{v}(r)$ is given by equation (2). A three-dimensional, nonsteady secondary flow (u, v, w) is superimposed on this basic flow, and the equations of motion are linearized. If it is assumed that the secondary motion also has rotational symmetry, the pressure does not appear in the tangential momentum equation

$$\frac{\partial v}{\partial t} + u\bar{\zeta} = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) v \quad (3)$$

where $\bar{\zeta} = dr\bar{v}/rdr$ is again the local axial vorticity in the basic flow. It is therefore convenient to eliminate the pressure entirely by combining the remaining two momentum equations into a single equation for the tangential component η of the secondary vorticity $(\xi, \eta, \zeta) = (-\partial v/\partial z, \partial u/\partial z - \partial w/\partial r, \partial v/r\partial r)$. The result is

$$\frac{\partial \eta}{\partial t} + 2\bar{\zeta}\bar{\omega} = \nu \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} + \frac{\partial^2}{\partial z^2} \right) \eta \quad (4)$$

where $\bar{\omega} = \bar{v}/r$ is again the local angular velocity of the basic flow. The two dynamical equations (3) and (4) are to be solved together with the continuity equation

$$\frac{1}{r} \frac{\partial ru}{\partial r} + \frac{\partial w}{\partial z} = 0 \quad (5)$$

Analysis

The material in the preceding introduction is not new and is included only to lay a foundation for a simplified analysis which will now be offered as a substitute for a formal solution of equations (3-5). The method is to replace the differential equations of the problem by algebraic equations, using uniformly valid estimates for the various dynamical elements of the motion. The starting point is the observation that the equations exhibit a natural division of the secondary flow into (a) a combined radial and axial part $u(r, z, t), w(r, z, t)$ in a meridional plane $\theta = \text{const}$,

and (b) a tangential part $v(r, z, t)$. The instability mechanism is contained in the coupling of these two secondary-flow components to each other and to the basic flow through the dynamical equations. In fact, this coupling implies a slight generalization of Rayleigh's original inviscid criterion. Steady flow or not, viscous flow or not, the two essential elements are clearly (a) finite vorticity in the basic flow ($\bar{\zeta} \neq 0$), without which the Coriolis term $u\bar{\zeta}$ could not couple the radial and tangential secondary flows in equation (3); and (b) finite curvature of the streamlines in the basic flow ($\bar{\omega} \neq 0$), without which the tangential vorticity-production mechanism represented by the term $2\bar{\zeta}\bar{\omega}$ would be inoperative in equation (4). The viscosity, on the other hand, is not an essential element unless the secondary motion is required to be steady, in which case the viscosity is purely stabilizing. It is this fact that makes it possible to estimate the effect of viscosity without solving the formal eigenvalue problem.

Consider a suitable dimensionless form of the tangential momentum equation (3). The ratio of Coriolis acceleration to tangential viscous force can be written as $-L^2\bar{\zeta}U/\nu V$, where L is a characteristic scale for the secondary flow; U and V are characteristic velocities for the radial (or axial) and tangential secondary flows, respectively; and $\bar{\zeta} < 0$ is the (constant) axial vorticity in the viscous basic flow. For steady motion, the terms in question must be in balance, both locally and in the large, and the parameter $-L^2\bar{\zeta}U/\nu V$ must therefore be of order unity.

A similar estimate can be made for the tangential vorticity equation (4). Again, in suitable dimensionless coordinates, the ratio of vorticity production to diffusion is $2L^2\bar{\Omega}V/\nu U$, where $\bar{\Omega}$ is some appropriate characteristic or mean value for the angular velocity $\bar{\omega}(r)$. This combination $2L^2\bar{\Omega}V/\nu U$ must also be of order unity for steady flow.

The principal dimensionless parameter of the problem now emerges on eliminating the ratio V/U between the two conditions just obtained.² When signs are taken into account, it is seen that the assumption of steady secondary flow requires

$$1 + 2 \frac{L^4\bar{\zeta}\bar{\Omega}}{\nu^2} = 0 \quad (6)$$

where the characteristic scale L and the characteristic angular velocity $\bar{\Omega}$ have yet to be precisely defined.³ Rayleigh's inviscid criterion, $\bar{\zeta}\bar{\omega} < 0$, has thus been replaced by a viscous criterion

$$\bar{\zeta}\bar{\Omega} = -\frac{\nu^2}{2L^4} \quad (7)$$

which reduces to the inviscid one in the limit $\nu \rightarrow 0$. The effect of viscosity is apparently to delay the onset of instability, as the speed of the basic motion increases, until the geometric mean of the two characteristic rotation times $1/\bar{\zeta}$ and $1/\bar{\Omega}$ decreases to a value of the same order as the characteristic diffusion time L^2/ν . Taylor's experimental results in Fig. 1 confirm that the critical

² If the length L is eliminated instead, it is found that the ratio of tangential to radial (or axial) velocity perturbation is $V/U = (-\bar{\zeta}/2\bar{\Omega})^{1/2} = (-A/\bar{\Omega})^{1/2}$. This ratio can be small (when the flow is nearly irrotational) or large (for strong opposite rotation). Possible interpretations in terms of helix angle, or frequency ratio, or relative energy level, are so far unexplored.

³ The particular combination of variables in equation (6) is frequently called the Taylor number, although different authors have used different quantities for L and $\bar{\Omega}$, much as different authors might use Reynolds numbers based on different lengths or velocities to describe the same boundary-layer flow. The square root of this dimensionless combination is also sometimes called the Taylor number. This practice, however, tends to obscure the physical meaning of the parameter, a meaning which only becomes clear when two dynamical processes are considered to be simultaneously in balance; see the remarks following equation (7). At one time, the term Taylor number was also used to refer to a combination $\bar{\Omega}L^2/\nu$ which measures the ratio of Coriolis accelerations to viscous forces for equations written in a coordinate system rotating with a steady angular velocity $\bar{\Omega}$. This usage, however, reflects Taylor's contributions to a quite different problem in the mechanics of rotating fluids, and current practice in geophysics is to call the reciprocal quantity $\nu/\bar{\Omega}L^2$ the Ekman number.

rotation rates $\bar{\zeta}$ and $\bar{\Omega}$ decrease as the scale L , i.e., the gap, increases for fixed ν .

Numerical estimates for L and $\bar{\Omega}$ can be based on experiments, or on numerical solutions of the formal eigenvalue problem, or even on nothing more than a rough guess as to the nature of the secondary motion.⁴ For cylinders rotating in the same direction, for example, theory and experiment agree that the secondary motion is a cellular vortex structure which is periodic in the z -direction. The cells occupy the whole of the fluid and are very nearly square in cross section. Thus the length L should be of the same order as, but smaller than, the gap between the cylinders. Moreover, the two angular velocities ω_0 and ω_i have at least the same sign, so that $\bar{\Omega}$ in equation (6) might be taken, say, as the arithmetic mean of ω_0 and ω_i , or as the value of $\bar{\omega}$ when r is the arithmetic or geometric mean of r_0 and r_i . When the gap is sufficiently small, these definitions become equivalent, and it can be concluded for this case that the parameter

$$T = -\frac{4A(r_0 - r_i)^4}{\nu^2} \frac{(\omega_0 + \omega_i)}{2} \quad (8)$$

should be nearly constant on the stability boundary. In the narrow-gap limit, the constant value of T (more properly, the minimum of T with respect to variations in axial wave number) is known from numerous theoretical calculations to be very nearly 1708; e.g., Chandrasekhar [5]. If the estimate $\bar{\Omega} = (\omega_0 + \omega_i)/2$ is adopted, it then follows on comparing equations (6) and (8) that the ratio $L/(r_0 - r_i)$, which is to say, the fraction of a narrow gap corresponding to the viscous or dissipative scale in the secondary flow, should be taken as $(1708)^{-1/4}$, or about $1/6$ to $1/7$.

Similar considerations apply for the case of opposite rotation, if it is assumed that the secondary motion occupies only the region which is unstable according to Rayleigh's criterion; i.e., the region between the inner cylinder and the nodal radius $r = r_c = (-B/A)^{1/2}$. Under these conditions, it can be expected, taking $\bar{\Omega}$ in equation (6) as proportional to ω_i , that some combination such as

$$T' = -\frac{4A(r_c - r_i)^4 \omega_i}{\nu^2} \quad (9)$$

will be essentially constant on the stability boundary.⁵ This expectation is also confirmed by detailed calculations. For example, Harris and Reid [8] find very nearly $T' = 1179$ for the limiting case of a narrow gap. With $\bar{\Omega} = \omega_i/2$, the ratio $L/(r_c - r_i)$ should therefore be taken as $(1179/2)^{-1/4}$, or about $1/5$.

There is no reason to suppose that the arguments developed here are limited to the case of a narrow gap, as long as proper estimates for L and $\bar{\Omega}$ can be found. For a finite gap, it seems likely that at least the representation for $\bar{\Omega}$ should be modified, perhaps by taking $\bar{\Omega}$ as an integral mean $(1/s) \int \bar{\omega} ds$, say, over the part of the annulus which is unstable by the inviscid criterion.⁶ This particular definition, together with equation (2), then implies for cylinders rotating in the same direction

$$\bar{\Omega} = A + \frac{2B}{r_i^2} \frac{\ln \left(\frac{r_0}{r_i} \right)}{\left(\frac{r_0^2}{r_i^2} - 1 \right)}, \quad L = \frac{(r_0 - r_i)}{(1708)^{1/4}} \quad (10)$$

⁴ For example, the methods of this paper have been used to predict the occurrence of cellular instability in a four-roller apparatus of the type originally used by Taylor [4] to produce a uniform longitudinal rate of strain in a viscous fluid.

⁵ The parameter T' was first introduced by Meksyn [6], who obtained the preliminary numerical value $T' = 1133$. An interesting discussion of this parameter appears in the paper by Donnelly and Fultz [7], who show from the condition $T' = \text{const}$ that $(\omega_0 r_0^2/\nu)$ varies as $(\omega_i r_i^2/\nu)^{1/3}$ for very strong opposite rotation.

⁶ Methods for averaging the parameters in the Taylor problem seem to be in the air. The scheme proposed by Walowit, et al. [9], and the one proposed here [cf., my equation (12) and their equation (30)] were apparently developed independently at about the same time.

and for opposite rotation

$$\bar{\Omega} = A + \frac{B}{r_i^2} \frac{\ln \left(-\frac{B}{Ar_i^2} \right)}{\left(1 + \frac{B}{Ar_i^2} \right)}, \quad L = \frac{r_i \left(\sqrt{\frac{-B}{Ar_i^2}} - 1 \right)}{\left(\frac{1179}{2} \right)^{1/4}} \quad (11)$$

where the original estimates for L have been retained. The stability boundary defined by equation (6) then becomes, for rotation in the same direction

$$1708 + 4 \left[\frac{A(r_0 - r_i)^2}{\nu} \right]^2 \left[1 + \frac{2B}{Ar_i^2} \frac{\ln \left(\frac{r_0}{r_i} \right)}{\left(\frac{r_0^2}{r_i^2} - 1 \right)} \right] = 0 \quad (12)$$

and for opposite rotation

$$1179 + 8 \left[\frac{Ar_i^2}{\nu} \right]^2 \left(\sqrt{\frac{-B}{Ar_i^2}} - 1 \right)^4 \left[1 + \frac{\ln \left(\frac{-B}{Ar_i^2} \right)}{\left(1 + \frac{B}{Ar_i^2} \right)} \right] = 0 \quad (13)$$

Discussion

The last two equations contain a main result of the present analysis; this is the suggestion that *the effect of geometry can be suppressed by a proper choice of variables*. The quantities usually measured in an experiment are the viscosity ν , the radii r_1 and r_0 , and the angular velocities ω_i and ω_0 . Most writers have therefore chosen as dimensionless variables the ratio r_i/r_0 , together with either (a) two Reynolds numbers $\omega_0 r_0^2/\nu$ and $\omega_i r_i/\nu$ or (b) the ratio ω_0/ω_i and a suitable Taylor number. The first choice is usually more convenient in experimental work (cf. Fig. 1), while the second choice is popular among theoreticians. However, equation (13) clearly involves only *two* variables, which can conveniently be taken⁷ as $-Ar_i^2/B$ and B/ν , while equation (12) can be rewritten in terms of $(-Ar_i^2/B)(r_0^2/r_i^2 - 1)/2 \ln(r_0/r_i)$ and $(B/\nu)(r_0/r_i - 1) \ln(r_0/r_i)/(r_0/r_i + 1)$.

For rotation of the two cylinders in the same direction and in opposite directions, respectively, the formulas (12) and (13) are compared in Figs. 2 and 3 with some experimental observations of the Taylor boundary. Inasmuch as several special cases of the formulas have been thoroughly verified by other writers, the agreement is not unexpected.⁸ The collapse of the data onto a single curve is most striking for the case of opposite rotation in Fig. 3. This point is reinforced by Fig. 4, which shows the

⁷ The inviscid stability boundaries are originally defined by $-Ar_i^2/\nu = 0$ ($v_0 r_0 = v_i r_i$, or $A = 0$) and by $-Ar_i^2/\nu = B/\nu$ ($\omega_i = 0$). In the coordinates in Figs. 3 and 4, these become $-Ar_i^2/B = 0$ and $-Ar_i^2/B = 1$. The choice of coordinates thus emphasizes conditions for a wide gap and/or a small Reynolds number; so does the use of a logarithmic scale for the abscissa B/ν . One result is a clearer view of the way in which the inviscid stability boundaries serve as asymptotes for the viscous one.

⁸ Lewis [10] observed a definite hysteresis in the stability boundary for increasing and decreasing speed. The reason for this hysteresis is unclear; it is the mean experimental boundary which is plotted in the figure and which agrees best with the formulas of this paper. Concerning a serious discrepancy between equation (13) and the measurements by Donnelly and Fultz [7] for strong opposite rotation (fluid with $\nu \sim 0.009 \text{ cm}^2/\text{sec}$), note their statement (Donnelly and Fultz [7, p. 119]) that convection currents due to frictional heating may have caused the basic flow to be abnormally stable. The data in question are included in Fig. 2 but are omitted for clarity in Fig. 3. Similarly, certain of the measurements by Coles (rough cylinders) and by Taylor (widest gap) for rotation in the same direction are included in Fig. 3 but are omitted in Fig. 2 in order to show equation (12) in the clearest light. Taylor [3, p. 336] comments that the fluid temperature was not under good control for his measurements with the widest gap. In the case of my own measurements, the disagreement is almost certainly due to irregularity of the cylindrical surfaces. Some improvement has been obtained by using the largest clearance to compute the radius ratio, but the remaining differences seem to be irreducible.

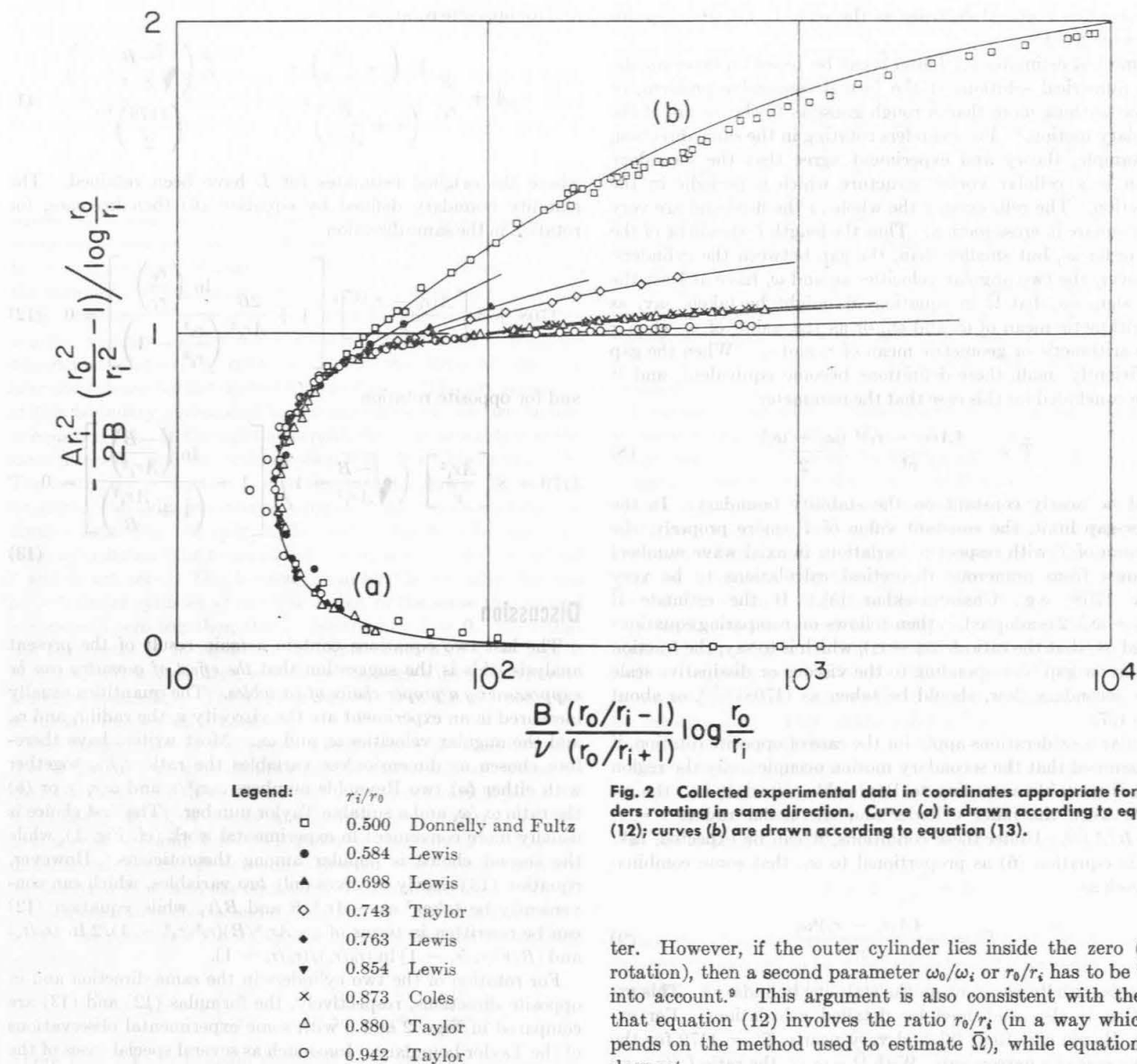


Fig. 2 Collected experimental data in coordinates appropriate for cylinders rotating in same direction. Curve (a) is drawn according to equation (12); curves (b) are drawn according to equation (13).

theoretical Taylor boundary as computed by Sparrow, et al. [11], for values of r_i/r_0 as small as 0.1. The "wide-gap" curve in Fig. 4 corresponds to equation (13); the "narrow-gap" curve is similar except that the integral estimate for $\bar{\Omega}$ in equation (11) is replaced by the estimate $\bar{\Omega} = \omega_i/2 = (A + B/r_i^2)/2$. Although the narrow-gap formula becomes inaccurate for r_i/r_0 less than about 0.85, and the wide-gap formula becomes inaccurate for r_i/r_0 less than about 0.35, the exact values in Fig. 4 continue to define essentially a single line down to $r_i/r_0 = 0.1$. In other words, for large enough $-\omega_0/\omega_i$, the stability boundary for opposite rotation is a functional of, say, $-Ar_i^2/\nu$ and B/ν alone, whatever the value of r_i/r_0 .

An argument can be found to explain the last result and, at the same time, suggest that it should continue to hold for arbitrarily small values of r_i/r_0 . Suppose that the basic velocity profile $\bar{v}(r)$ of equation (2) is normalized to express \bar{v}/v_i (or $\bar{\omega}/\omega_i$) as a one-parameter function of r/r_i . The parameter is $-Ar_i^2/B$, and this is known to lie between 0 and 1 whenever the flow is unstable according to the inviscid criterion. Now \bar{v}/v_i has a zero at $(-B/Ar_i^2)^{1/2}$, which lies in the interval $1 < r/r_i < \infty$. Beyond this zero, the flow is stable in the inviscid sense. If the outer cylinder itself lies far beyond this zero (strong opposite rotation), then its actual position is almost irrelevant and, for practical purposes, the instability problem is defined by the single param-

eter. However, if the outer cylinder lies inside the zero (same rotation), then a second parameter ω_0/ω_i or r_0/r_i has to be taken into account.⁹ This argument is also consistent with the fact that equation (12) involves the ratio r_0/r_i (in a way which depends on the method used to estimate $\bar{\Omega}$), while equation (13) does not.

Finally, one further comment should be made about the role of the quantities A and B as natural coordinates. It is natural to characterize a (statistically) steady Couette flow by the torque $M = 2\pi r^2 \tau$ at either of the two cylindrical walls. But $\tau = \mu r \partial(\bar{v}/r)/\partial r$ for circulatory flow, and hence $|\tau r^2| = 2\mu B$ if $\bar{v} = Ar + B/r$. On the instability boundary, therefore,

$$\frac{M}{4\pi\mu\nu} = \frac{B}{\nu} \quad (14)$$

A second natural parameter is the mean axial vorticity, computed by dividing the area of the annulus into the difference between the circulations $\Gamma = 2\pi r v$ at the two walls. Since the mean and local vorticities for the particular flow $\bar{v} = Ar + B/r$ are both given by $\bar{\zeta} = 2A$, it follows that

⁹ In theoretical work on the Taylor eigenvalue problem, it is a major difficulty that a variable coefficient $\bar{\omega}(r)$ appears in equation (4). For the case of cylinders rotating in the same direction, the approximation is sometimes made (e.g., Lin [12, p. 19]) of replacing this variable coefficient by a constant mean value such as $\bar{\Omega} = (\omega_0 + \omega_i)/2$. However, since it is really the circulation rather than the angular velocity which is nearly constant, the approximation becomes increasingly poor as the gap becomes wider. The results of this paper favor using $\bar{\Omega}$ from equation (10). For strong opposite rotation it may be sufficient for most purposes to compute the eigenfunctions only once for each value of B/ν rather than once for each combination of B/ν and r_i/r_0 .

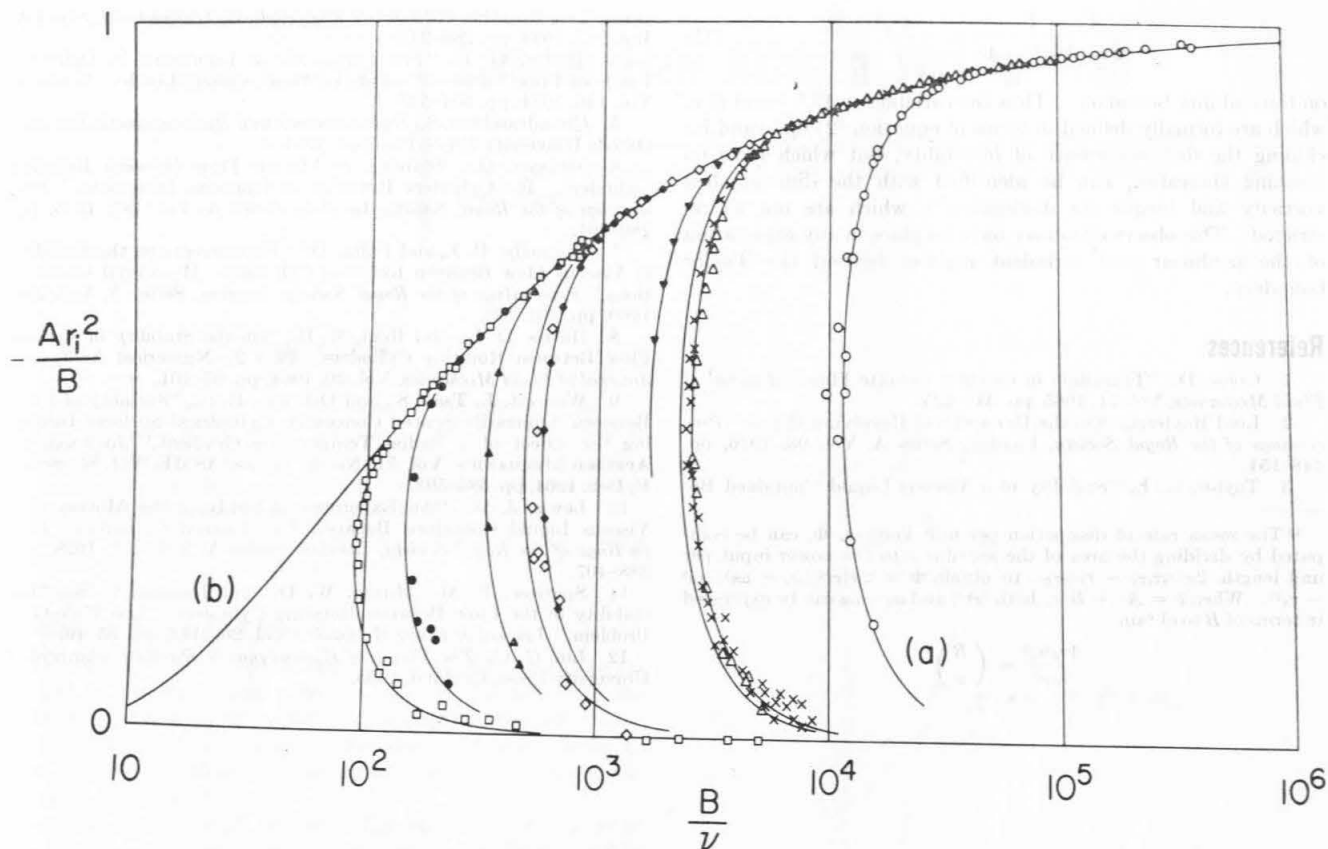


Fig. 3 Collected experimental data in coordinates appropriate for cylinders rotating in opposite directions. Curves (a) are drawn according to equation (12); curve (b) is drawn according to equation (13). For legend, see Fig. 2.

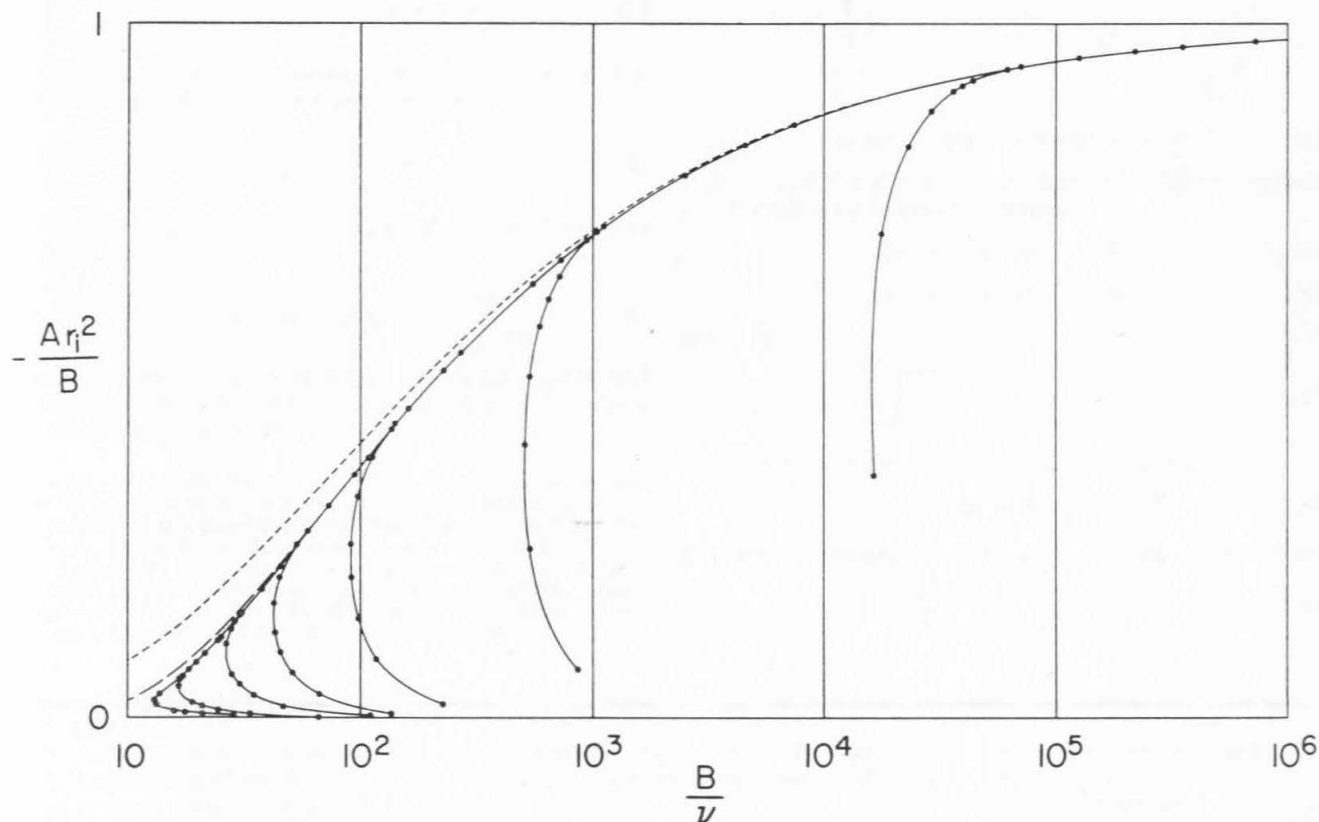


Fig. 4 Theoretical results of Sparrow, et al. [11], in coordinates appropriate for cylinders rotating in opposite directions. Solid lines are for fixed values of r_i/r_0 , with r_i/r_0 increasing from 0.10 at the left to 0.95 at the right. Dashed line is asymptotic wide-gap formula as given by equation (13). Dotted line is corresponding narrow-gap formula (see text).

$$\frac{\Gamma_0 - \Gamma_i}{2\pi\nu(r_0^2/r_i^2 - 1)} = -\frac{Ar_i^2}{\nu} \quad (15)$$

on the stability boundary. Thus the variables $-Ar_i^2/\nu$ and B/ν , which are formally defined in terms of equation (2) up to and including the first occurrence of instability, but which have no meaning thereafter, can be identified with the dimensionless vorticity and torque (or dissipation¹⁰), which are not so restricted. This observation may have its place in any exploration of the nonlinear and turbulent regimes beyond the Taylor boundary.

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¹⁰ The mean rate of dissipation per unit volume, Φ , can be computed by dividing the area of the annulus into the power input per unit length, $2\pi(r_i v_i r_i - r_0 v_0 r_0)$, to obtain $\Phi = 2\pi[r_i^2(\omega_i - \omega_0)/(r_0^2 - r_i^2)]$. When $\bar{v} = Ar + B/r$, both $|rr^2|$ and $\omega_i - \omega_0$ can be expressed in terms of B to obtain

$$\frac{\Phi r_0^2 r_i^2}{4\mu\nu^2} = \left(\frac{B}{\nu}\right)^2$$

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